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# A remark on the 2D-Euler equation (Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics)

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## A remark on the 2D-Euler equation

In this paper we revisit the initial value problem for the 2D-Euler equation on a bounded domain. The main object is to streamline the proof of the global existence and uniqueness of a classical solution, given in the old paper [K], although there is nothing essentially new. In particular we use the vorticity  $\zeta = \partial \wedge u (= \text{curl}(u))$  as a basic ingredient of the theory. However, instead of assuming that the initial velocity  $a$  is  $C^{1+\theta}$  as in [K], we simply assume that  $\alpha = \partial \wedge a$  is  $C$  and construct a unique *weak* solution  $u(t)$  in  $\widehat{L}$ , to be defined below. Afterwards it is shown that if  $a \in C^{1+\theta}$  then  $u(t) \in C^{1+\theta}$ . Almost all the necessary material is in [K]; the change is only in the order of their arrangement. Naturally we follow the notation of [K] as much as possible.

As in [K], we consider a bounded domain  $\Omega \subset \mathbf{R}^2$ ; for simplicity we assume that  $\Omega$  is smooth and simply connected, and that there is no external force. (The modification necessary for a multiply connected  $\Omega$  will be commented on later.) Moreover, for notational convenience we assume that  $\Omega$  is closed. (If necessary we use  $\Omega^\circ$  to denote the interior of  $\Omega$ .)

We denote by  $\|\cdot\|$  the  $C(\Omega)$ -norm, indiscriminately for scalar or vector valued functions.  $\widehat{L}(\Omega; \mathbf{R}^2)$  is the set of all vector valued functions on  $\Omega$  such that

$$f \in W^{1,p}(\Omega; \mathbf{R}^2) \quad \text{for } 1 < p < \infty, \quad \text{and}$$

$$|f(x) - f(y)| \leq \text{const.} \omega(|x - y|), \quad x, y \in \Omega,$$

where  $\omega(s) = s(1 + \log^+(1/s))$ . The associated norm is denoted by  $\|f\|_{\text{ql}}$ .

The initial value problem for the Euler equation is given by

$$\partial_t u + \partial \cdot (uu) + \partial p = 0, \quad \partial \cdot u = 0, \quad u(0) = a. \quad (1)$$

Here  $uu$  is a tensor with  $jk$  component  $u_j u_k$ ;  $\partial \cdot (uu)$  is a vector with  $k$  component  $\partial_j (u_j u_k)$ ;  $\partial \cdot u = \text{div}(u) = \partial_j u_j$ . (Summation convention is used throughout.)

**Theorem I.** Let  $\partial \wedge a \in C(\Omega; \mathbf{R})$  and  $T > 0$ . Then there is a unique weak solution  $\{u, p\}$  to (1) such that

$$u \in C(I; \widehat{L}(\Omega; \mathbf{R}^2)), \quad \partial p \in \quad, \quad I = [0, T]. \quad (2)$$

If in particular  $\partial \wedge a \in C^\theta(\Omega; \mathbf{R})$  for some  $\theta \in (0, 1)$ , then  $\{u, p\}$  is a classical solution with the properties

$$u \in C(I; C^1(\Omega; \mathbf{R}^2)) \cap B(I; C^{1+\theta}(\Omega; \mathbf{R}^2)), \quad \partial_t u \in C(I; C(\Omega; \mathbf{R}^2)), \quad \partial p \in$$

where  $B$  denotes the class of bounded functions.

For the proof we introduce the (scalar) vorticity

$$\zeta = \partial \wedge u = (\partial_1 u_2 - \partial_2 u_1). \quad (4)$$

As is well known  $\zeta$  should satisfy the *vorticity equation*, which is a system consisting of (4) and

$$\partial_t \zeta + \partial \cdot (u \zeta) = 0, \quad \zeta(0) = \alpha = \partial \wedge a. \quad (5)$$

Our plan is to start with a function  $\varphi$  in a certain subset  $S$  of  $C(Q)$ , where  $Q = I \times \Omega$ , and determine  $u \in C(Q)$ , which are q.L. in  $x$ , such that  $\partial \wedge u = \varphi$ . We then solve (4) for  $\zeta$ , which is shown to be in a certain compact subset of  $S$ . Furthermore, we show that the map  $\varphi \mapsto \zeta$  is continuous in  $C(Q)$ . A fixed point of the map, which exists by the Schauder fixed point theorem, gives a solution of the vorticity equation.  $u$  will then be shown to be the unique solution of (1) together with a certain gradient  $\partial p$ .

**Lemma 1.** For each  $\varphi \in C(Q; \mathbf{R})$ , there is a unique  $u \in C(I; \hat{L})$  such that

$$\begin{aligned} \partial \cdot u(t) = 0 \quad \text{and} \quad \partial \wedge u(t) = \varphi(t) \quad \text{on } \Omega, \quad \|u(t)\| = 0 \quad \text{on } b\Omega, \\ \|u(t)\|_L \leq c \|\varphi(t)\|, \quad t \in I, \end{aligned} \quad (6)$$

where  $c$  is a constant depending only on  $\Omega$ .

*Proof.* This follows immediately from [K, Lemma x.x]; note that  $C(Q; \mathbf{R}) = C(I; C(\Omega))$ .

**Lemma 2.** Let  $u \in C(Q; \mathbf{R}^2)$  such that  $u(t) \in \hat{L}(\Omega)$ ,  $\partial \cdot u(t) = 0$  on  $\Omega$  and  $\nu \cdot u(t) = 0$  on  $b\Omega$ . Then the ordinary differential equation  $dx/dt = u(t, x)$  is uniquely solvable for any initial time  $s \in I$  and any initial condition  $x(s) = y \in \Omega$ , with the solution (characteristic function)  $x = \Phi_{t,s}(y) \in \Omega$  existing for all  $t \in I$ . The map  $\Phi : t, s, y \mapsto x$  is continuous in the three variables. For fixed  $t, s$ , it is a homeomorphism of  $\Omega$  onto itself, satisfying the chain rule  $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$ .

*Proof.* The existence of the solution for all  $t, s$  is due to the fact that  $\partial \cdot u = 0$  and  $\nu \cdot u = 0$  (see [K]). The uniqueness follows from the theorem of Osgood, since  $1/\omega(r)$  is not integrable near  $r = 0$ . For the continuity properties, see e.g. [H].

**Lemma 3.** Let  $u_n$ ,  $n = 1, 2, \dots$ , be a sequence of functions satisfying the assumptions of Lemma 2, with the associated map  $\Phi_n$ . Moreover, assume that  $u_n \rightarrow u$  in  $C(Q; \mathbf{R}^2)$ . Then  $\Phi_n \rightarrow \Phi$  in  $C(Q; \mathbf{R}^2)$ .

*Proof.* This is a continuous dependence theorem for the characteristic function. Usually it is stated as continuous dependence on a auxiliary continuous parameter  $\mu$  (see e.g.[H]), but there is no difference in the proof when  $\mu$  is replaced by a discrete parameter  $n$ .

**Lemma 4** The homeomorphisms  $\Phi_{t,s}$  are measure preserving.

*Proof.* Approximate  $u$  in  $\hat{L}$  by  $C^1$  functions, for which  $\Phi$  becomes  $C^1$  in all three variables and the result is classical (see e.g.[H]). The required result follows on passing to the limit using Lemma 3.

**Lemma 5**  $\Phi_{t,s}(y)$  is uniformly Hölder continuous in the three variables for  $t, s \in I, y \in \Omega$ .

*Proof.* The result is due to the quasi-Lipashitzian property of  $u$ , see [K], Lemma x.x. The Hölder exponent may be very small when  $T$  is large.

**Lemma 6** Let  $u$  be as in Lemma 2. Then the linearized vorticity equation (2) has a weak solution  $\zeta$  given by

$$\zeta(t) = \alpha \circ \Phi_{0,t}, \quad t \in I. \quad (7)$$

*Proof.* This is well known for a classical solution if  $u$  and  $\alpha$  were  $C^1$ . As it is, it requires a proof. Obviously (7) satisfies  $\zeta(0) = \alpha$ , since  $\Phi_{0,0}$  is the identity on  $\Omega$ . Thus it suffices to show that for any smooth scalar function  $\chi$  on  $Q$ , one has

$$\partial_t \langle \zeta, \chi \rangle = \langle \zeta u, \partial \chi \rangle = \langle \zeta, u \cdot \partial \chi \rangle, \quad (8)$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product on  $\Omega$  for scalar or vector valued functions. In view of (7) and the measure preserving property of the map  $\Phi_{t,s}$ , (8) is equivalent to

$$\partial_t \langle \alpha, \chi \circ \Phi_{t,0} \rangle = \langle \alpha, (u \cdot \partial \chi) \circ \Phi_{t,0} \rangle; \quad (9)$$

note that  $\Phi_{t,0}$  is the inverse map of  $\Phi_{0,t}$ . Here the left member equals

$$\begin{aligned} & \langle \alpha(x), \partial_t \chi(\Phi_{t,0}(x)) \rangle = \langle \alpha(x), \partial \chi(\Phi_{t,0}(x)) \cdot \partial_t \Phi_{t,0}(x) \rangle \\ & = \langle \alpha(x), \partial \chi(\Phi_{t,0}(x)) \cdot u(t, \Phi_{t,0}(x)) \rangle \end{aligned}$$

which is the right member of (9), q.e.d.

*Remark.* It appears that Lemma 4 is nontrivial; it would be hard to prove it without the condition  $\partial \cdot u = 0$ , which implies the measure preserving property.

**Lemma 7** There is  $u \in C(I; \hat{L}(\Omega; \mathbf{R}^2))$  such that  $\zeta = \partial \wedge u$  is in  $C(Q; \mathbf{R})$  and is a weak solution of the vorticity equation ( ).

*Proof.* Let  $\alpha \in C(\Omega)$  be fixed. Let  $S$  be the ball in  $C(Q)$  with center 0 and radius  $\|\alpha\|$ . For each  $\varphi \in S$ , construct  $u$  and then  $\zeta$  according to Lemmas 2 and 5. Then it is obvious that  $\|\zeta\| \leq \|\alpha\|$ , hence  $\zeta \in S$ . Thus the map  $F : \varphi \mapsto \zeta$  sends  $S$  into itself.  $F$  is continuous in the topology of  $C(Q)$ , as is seen from Lemmas 2,3. Moreover, the range of  $F$  is compact in  $C(Q)$ , since  $\zeta(t, x) = \alpha(\Phi_{0,t}(x))$ , where  $\alpha \in C(\Omega)$  is fixed and  $\Phi_{0,t}(x)$  is uniformly Hölder continuous in  $t, x$  by Lemma 5. It follows from Schauder's fixed point theorem that  $F$  has a fixed point  $\zeta$ , which is a solution of the vorticity equation.